Schwartz
3.6 (a)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2 - A_\mu J_\mu$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[ -\frac{1}{4} F_{\mu\nu}^2 \right] = \partial_\mu F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A_\nu - J_\nu$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial(A_\nu)} = 0$$

$$\Rightarrow \boxed{\partial_\mu F_{\mu\nu} = J_\nu - m^2 A_\nu \quad \text{(EQM)}}$$

Taking $\partial_\nu$ of this equation gives

$$\partial_\nu \partial_\mu F_{\mu\nu} = \partial_\nu J_\nu - m^2 \partial_\nu A_\nu$$

By assumption $\partial_\nu J_\nu = 0$, then expand $F_{\mu\nu}$ gives

$$\partial_\nu \partial_\mu [\partial_\mu A_\nu - \partial_\nu A_\mu] = -m^2 \partial_\nu A_\nu$$

$$\Box \partial_\nu A_\nu - \Box \partial_\mu A_\mu = -m^2 \partial_\nu A_\nu$$

$$\boxed{0 = \partial_\nu A_\nu}$$

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Schwartz
3.6 (c)

$$A_0(r) = \frac{e}{4\pi^2 ir} \int_{-\infty}^{\infty} \frac{k\,dk}{k^2+m^2} e^{ikr}$$

$$\frac{k e^{ikr}}{k^2+m^2} = k e^{ikr} \frac{1}{(k+im)(k-im)}$$

There are two singularities, we close contour from above, meaning we take the $+im$ as the singularity.

Expanding $k e^{ikr} \frac{1}{(k+im)(k-im)}$ in Laurent series around $k_0 = im$

$$k e^{i[k-im+im]r} \frac{1}{(k+im)(k-im)}$$

$$= k\, e^{i(k-im)r} e^{i(im)r} \frac{1}{(k+im)(k-im)}$$

exponentials have only positive powers of $(k-im)$, so the coefficient for the $(k-im)^{-1}$ term is

$$\frac{k e^{-mr}}{k+im}.$$ this is the residue.

Evaluating the residue at $k_0 = im$ gives

$$\text{Res}[k_0 = im] = \frac{im\, e^{-mr}}{2im}$$

$$\Rightarrow \quad 2\pi i \,\text{Res}[k_0 = im] = \frac{\cancel{2}\pi \cancel{(i)}(i)\cancel{m}\, e^{-mr}}{\cancel{2im}}$$

$$= \pi i\, e^{-mr} = \int_{-\infty}^{\infty} \frac{k\,dk}{k^2+m^2}\, e^{ikr}$$

$$\Rightarrow \quad A_0(r) = \frac{e}{4\pi^{\cancel{2}}\cancel{i}\, r}\, \cancel{\pi}\cancel{i}\, e^{-mr}$$

$$= \boxed{\frac{e}{4\pi r} \exp[-mr]}$$

(d) The $\lim_{m \to 0} A_0$ is obviously Coulomb Potential soln.